A New Proof of the Formulae for Right-Angled Spherical Triangles.

By Professor JOHN JACK.

It is assumed that the sines of the sides are proportional to the sines of the opposite angles.

ACB (Fig. 9) is a spherical $\triangle$, with C a right angle.

Produce AC, AB to D and E so that $AD = AE = \frac{\pi}{2}$.

Draw the great $\odot$ DEF and produce CB to meet it in F.

Then F is the pole of AD and A the pole of DF.

Then sides and angles of ABC are a b c A B

BEF are $\frac{\pi}{2} - A$ $\frac{\pi}{2} - c$ $\frac{\pi}{2} - a$ B $\frac{\pi}{2} - b$.

$$\therefore \quad \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{1} \quad - \quad - \quad - \quad \text{I.}$$

and
$$\frac{\sin(\frac{\pi}{2} - A)}{\sin B} = \frac{\sin(\frac{\pi}{2} - c)}{\sin(\frac{\pi}{2} - b)} = \frac{\sin(\frac{\pi}{2} - a)}{1}$$

that is
$$\frac{\cos A}{\sin B} = \frac{\cos c}{\cos b} = \frac{\cos a}{1} \quad - \quad - \quad - \quad \text{II.}$$

and $\therefore$
$$\frac{\cos B}{\sin A} = \frac{\cos c}{\cos a} = \frac{\cos b}{1} \quad - \quad - \quad - \quad \text{III.}$$

by interchange of a, A and b, B.

$\therefore$ $\left.\begin{aligned} \sin a &= \operatorname{sinc} \sin A \\ \sin b &= \operatorname{sinc} \sin B \end{aligned}\right\}$ from I. - - - 1.

and $\left.\begin{aligned} \cos A &= \cos a \sin B \\ \cos B &= \cos b \sin A \end{aligned}\right\}$ from II., III. - - 2.

and $\cos c = \cos a \,.\, \cos b$ from II. or III. - 3.

From 2 $\cos A \cos B = \cos a \cos b \sin A \sin B$

$$\therefore \quad \cot A \cot B = \cos a \cos b = \cos c \text{ by 3} \quad - \quad - \quad - \quad 4.$$

Again $\operatorname{sinc} = \frac{\sin b}{\sin B}$ by I.

$\operatorname{cosc} = \frac{\cos A \cos b}{\sin B}$ by II. Divide

$$\therefore \quad \tan c = \frac{\tan b}{\cos A} \qquad \left.\begin{aligned} \therefore \quad \tan b &= \tan c \cos A \\ \text{and } \tan a &= \tan c \cos B \end{aligned}\right\} \quad - \quad - \quad 5$$

Again $\sin a = \dfrac{\sin b \sin A}{\sin B}$ by I.

and $\cos a = \dfrac{\cos A}{\sin B}$ by II. Divide

$$\therefore \tan a = \tan A \sin b$$
$$\text{so } \tan b = \tan B \sin a \quad \Big\} \quad \ldots\ldots 6.$$

Note on Napier's Rules.

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Denote the parts b A c B a of $\triangle$ ABC (Fig. 9)

by 1 2 3 4 5

then the parts corresponding of the $\triangle$ BEF, namely,

$$\tfrac{\pi}{2} - c, \quad \text{B}, \quad \tfrac{\pi}{2} - a, \quad \tfrac{\pi}{2} - b, \quad \tfrac{\pi}{2} - \text{A}$$

will be denoted by $\tfrac{\pi}{2} - 3, \quad 4, \quad \tfrac{\pi}{2} - 5, \quad \tfrac{\pi}{2} - 1, \quad \tfrac{\pi}{2} - 2.$

Now a third $\triangle$ can similarly be derived from this second, a fourth from the third, and a fifth from the fourth. But when the process is applied to the fifth, the first $\triangle$ is obtained. Hence only 5 $\triangle$s can be obtained, which are the following :—

1	2	3	4	5
$\frac{\pi}{2} - 3$	4	$\frac{\pi}{2} - 5$	$\frac{\pi}{2} - 1$	$\frac{\pi}{2} - 2$
5	$\frac{\pi}{2} - 1$	2	3	$\frac{\pi}{2} - 4$
$\frac{\pi}{2} - 2$	3	4	$\frac{\pi}{2} - 5$	1
$\frac{\pi}{2} - 4$	$\frac{\pi}{2} - 5$	$\frac{\pi}{2} - 1$	2	$\frac{\pi}{2} - 3$
1	2	3	4	5

where the mid-column contains the hypotenuse, the two next to it contain the angles, and the extreme columns the sides of the several right-angled triangles.